

# Computing DM-decomposition of a partitioned matrix with rank-1 blocks

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## Abstract

In this paper, we develop a polynomial time algorithm to compute a Dulmage-Mendelsohn-type decomposition of a matrix partitioned into submatrices of rank at most 1.

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## 1 Introduction

The *Dulmage-Mendelsohn decomposition* (*DM-decomposition*) [2, 3] of an  $n \times m$  matrix  $A$  is a canonical block-triangulation under transformation

$$A \mapsto P^\top A Q$$

for permutation matrices  $P$  and  $Q$ ; see also [9, Section 4.3] and [10, Section 2.2.3]. The DM-decomposition exploits zero submatrices of the largest size ( $\geq \max\{n, m\}$ ), where the size is defined as the sum of numbers of rows and columns. Finding such a zero submatrix of  $A$  is nothing but the maximum stable set problem in the bipartite graph associated with the nonzero pattern of entries of  $A$ . The family of maximum stable sets is regarded as the minimizer set of a submodular function, and forms a distributive lattice. The DM-decomposition is obtained by arranging rows and columns with respect to a maximal chain of this distributive lattice.

The present paper addresses a generalization of the DM-decomposition considered by Ito, Iwata, and Murota [5]; see also [10, Section 4.8]. This generalization deals with a matrix  $A$  partitioned into submatrices in the following form:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1\nu} \\ A_{21} & A_{22} & \cdots & A_{2\nu} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\mu 1} & A_{\mu 2} & \cdots & A_{\mu \nu} \end{pmatrix},$$

where  $A_{\alpha\beta}$  is an  $n_\alpha \times m_\beta$  matrix for  $\alpha = 1, 2, \dots, \mu$ ,  $\beta = 1, 2, \dots, \nu$ . Such a matrix is called a *partitioned matrix of type*  $(n_1, n_2, \dots, n_\mu; m_1, m_2, \dots, m_\nu)$ . In this setting, an admissible transformation is  $A \mapsto$

$$P^\top \begin{pmatrix} E_1^\top & O & \cdots & O \\ O & E_2^\top & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & E_\mu^\top \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1\nu} \\ A_{21} & A_{22} & \cdots & A_{2\nu} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\mu 1} & A_{\mu 2} & \cdots & A_{\mu\nu} \end{pmatrix} \begin{pmatrix} F_1 & O & \cdots & O \\ O & F_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & F_\nu \end{pmatrix} Q,$$

where  $E_\alpha$  is a nonsingular  $n_\alpha \times n_\alpha$  matrix for  $\alpha = 1, 2, \dots, \mu$ ,  $F_\beta$  is a nonsingular  $m_\beta \times m_\beta$  matrix for  $\beta = 1, 2, \dots, \nu$ ,  $P$  and  $Q$  are permutation matrices of size  $n$  and  $m$ , respectively. Ito, Iwata, and Murota [5] showed the existence of a canonical block-triangulation under this transformation, which we call the *DM-decomposition*. This generalization of DM-decomposition is obtained from the minimizer set of a submodular function on a modular lattice.

Submodular optimization on modular lattices is an undeveloped area of combinatorial optimization, and has just been started [4, 8]. It is an open problem in [5, p. 1252] to design a polynomial time algorithm to compute DM-decomposition of partitioned matrices. Currently such an algorithm is known for very restricted classes of partitioned matrices. For a partitioned matrix of type  $(n, m)$ , the DM-decomposition is the rank normal form, and is computed via Gaussian elimination. For a partitioned matrix of type  $(1, 1, \dots, 1; 1, 1, \dots, 1)$ , the DM-decomposition coincides with the original one [2, 3], and is computed via bipartite matching algorithm. For a partitioned matrix of type  $(n_1, n_2, \dots, n_\mu; 1, 1, \dots, 1)$  (or  $(1, 1, \dots, 1; m_1, m_2, \dots, m_\nu)$ ), the DM-decomposition is the *combinatorial canonical form (CCF) of multilayered mixed matrix* [10, 11], and is computed via matroid union algorithm. To the best of the author's knowledge, the computational complexity of other cases is completely unknown.

The main result of this paper is a polynomial time algorithm of the DM-decomposition for a new class of partitioned matrices.

**Theorem 1.1.** *Let  $A = (A_{\alpha\beta})$  be a partitioned matrix. Suppose that each submatrix  $A_{\alpha\beta}$  has rank at most 1. Then the DM-decomposition of  $A$  is computed in polynomial time.*

The rest of the paper is organized as follows. In Section 2, we provide a necessary background for lattice and matroid. In Section 3, we formally introduce the DM-decomposition of partition matrices. Our view is different from Ito, Iwata, and Murota [5]. We view a partitioned matrix  $A$  as a bilinear form on vector space  $U \times V$ , and formulate the problem, called *maximum stable subspace problem (MSSP)*, of finding a vector subspace  $(X, Y)$  in a specified sublattice of all vector subspaces of  $U \times V$  such that  $A$  vanishes on  $X \times Y$  and  $\dim X + \dim Y$  is maximum. This vector-space generalization of the bipartite stable set problem seems interesting in its own right. Then the DM-decomposition is obtained by a maximal chain of the family of maximum stable subspaces. In Section 4, we deal with the special case where each submatrix has rank at most 1. We show that (MSSP) reduces to the *maximum independent matching problem* [1], which is a version of matroid intersection problem. From a maximum independent matching, the transformation matrices for the DM-decomposition are obtained in polynomial time.

## 2 Preliminaries

### 2.1 Lattice

A *lattice* is a partially ordered set (poset)  $\mathcal{L} = (\mathcal{L}, \preceq)$  such that every pair  $p, q$  of elements has meet  $p \wedge q$  (greatest common lower bound) and join  $p \vee q$  (lowest common upper bound). By  $p \prec q$  we mean  $p \preceq q$  and  $p \neq q$ . A sequence  $p_0 \prec p_1 \prec \cdots \prec p_k$  of pairwise comparable elements is called a *chain* from  $p_0$  to  $p_k$  where  $k$  is called the length. In this paper, we only consider lattices with the properties that both the minimum element and the maximum element exist and the length of any maximal chain from the minimum to the maximum is finite. The rank  $r(p)$  of element  $p$  is defined as the maximum length of a chain from the minimum to  $p$ .

A lattice  $\mathcal{L}$  is said to be *distributive* if  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$  and  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$  hold for every triple  $x, y, z$  of elements. A canonical example of a distributive lattice is the family  $\mathcal{J}(\mathcal{P})$  of all ideals of a (finite) poset  $\mathcal{P}$ , where an *ideal* of  $\mathcal{P}$  is a subset  $J \subseteq \mathcal{P}$  of elements with the property that  $p \preceq q \in J$  implies  $p \in J$ . The partial order of ideal family  $\mathcal{J}(\mathcal{P})$  is given by the inclusion order. The following is a simpler version of the Birkhoff representation theorem.

**Lemma 2.1.** *A lattice  $\mathcal{L}$  is distributive if and only if it is isomorphic to  $\mathcal{J}(\mathcal{P})$  for some poset  $\mathcal{P}$ .*

A lattice  $\mathcal{L}$  is called *modular* if for every triple  $x, a, b$  of elements with  $x \preceq b$ , it holds  $x \wedge (a \vee b) = (x \vee a) \wedge b$ . It is known that a modular lattice is exactly such a lattice that satisfies

$$r(p) + r(q) = r(p \wedge q) + r(p \vee q) \quad (p, q \in \mathcal{L}).$$

A canonical example of a modular lattice is the family  $\mathcal{U}$  of all subspaces of a vector space  $U$ , where the partial order is the inclusion order. For two subspaces  $X, Y$ , the meet  $X \wedge Y$  is equal to the intersection  $X \cap Y$ , and the join  $X \vee Y$  is equal to the sum  $X + Y$ . The rank of  $X$  is equal to the dimension  $\dim X$ . The following equality of dimension is well-known:

$$\dim X + \dim Y = \dim(X \cap Y) + \dim(X + Y) \quad (X, Y \in \mathcal{U}). \quad (2.1)$$

### 2.2 Matroid

A *matroid*  $\mathbf{M} = (V, \mathcal{I})$  is a pair of a finite set  $V$  and a family  $\mathcal{I}$  of subsets of  $V$  such that  $\emptyset \in \mathcal{I}$  and for  $I, I' \in \mathcal{I}$  with  $|I| < |I'|$  there is  $e \in I' \setminus I$  such that  $I \cup \{e\} \in \mathcal{I}$ , where  $V$  is called the *ground set* and a member of  $\mathcal{I}$  is called an *independent set*. The *rank function*  $\rho$  is defined by  $\rho(X) := \max\{|I| \mid I \in \mathcal{I} : I \subseteq X\}$ . The *closure operator*  $\text{cl}$  is defined by  $\text{cl}(X) = \{e \in V \mid \rho(\{e\} \cup X) = \rho(X)\}$ . The *direct sum* of two matroids  $\mathbf{M} = (V, \mathcal{I})$  and  $\mathbf{M}' = (V', \mathcal{I}')$  is the matroid such that the ground set is the disjoint union  $V \cup V'$  and the independent sets are  $I \cup I'$  for  $I \in \mathcal{I}$  and  $I' \in \mathcal{I}'$ .

Let  $U$  be a finite-dimensional vector space. For a finite set  $V$  of vectors in  $U$  and the family  $\mathcal{I}$  of all linearly independent subsets of  $V$ , a pair  $(V, \mathcal{I})$  is a matroid. We will use a dual construction. Let  $\Pi$  be a finite subset of hyperplanes ( $(\dim U - 1)$ -dimensional subspaces) of  $U$  and let  $\mathcal{I}$  be the set of all subsets  $H \subseteq \Pi$  such that  $|H|$  is equal to  $\dim U$  minus the dimension of the intersection of hyperplanes in  $H$ . Then  $(\Pi, \mathcal{I})$  is a matroid to be denoted by  $\mathbf{M}(\Pi)$ .

**Independent matching problem.** Following [10, Section 2.3.5], we introduce the independent matching problem. Let  $G = (V^+, V^-, E)$  be a bipartite graph on vertex set  $V^+ \cup V^-$  ( $V^+ \cap V^- = \emptyset$ ) and edge set  $E \subseteq V^+ \times V^-$ . Let  $\mathbf{M}^+$  be a matroid on ground set  $V^+$ , and  $\mathbf{M}^-$  be a matroid on ground set  $V^-$ . The rank functions of  $\mathbf{M}^+$  and  $\mathbf{M}^-$  are denoted by  $\rho^+$  and  $\rho^-$ , respectively. The closure operators of  $\mathbf{M}^+$  and  $\mathbf{M}^-$  are denoted by  $\text{cl}^+$  and  $\text{cl}^-$ , respectively. For an edge subset  $F$ , let  $\partial^+ F$  denote the set of vertices in  $V^+$  incident to  $F$ , and let  $\partial^- F$  denote the set of vertices in  $V^-$  incident to  $F$ . A *matching* is an edge subset  $M$  with  $|M| = |\partial^+ M| = |\partial^- M|$ . A matching  $M$  is said to be *independent* if  $\partial^+ M$  is independent in  $\mathbf{M}^+$  and  $\partial^- M$  is independent in  $\mathbf{M}^-$ . The *independent matching problem* on  $(\mathbf{M}^+, \mathbf{M}^-, G)$  is the problem of finding a matching  $M$  of the maximum size. The following min-max is a reformulation of Edmonds' matroid intersection theorem, which was obtained by Brualdi [1]:

**Theorem 2.2** ([1]; see [10, Theorem 2.3.27]). *The maximum size of an independent matching is equal to the minimum of  $\rho^+(H) + \rho^-(K)$  over all covers  $(H, K)$ .*

Here a *cover*  $(H, K)$  is a pair of  $H \subseteq V^+$  and  $K \subseteq V^-$  such that every edge meets  $H \cup K$ . A cover  $(H, K)$  attaining the minimum of  $\rho^+(H) + \rho^-(K)$  is said to be *minimum*.

We will use an algorithm to solve the independent matching problem and related structures. For an independent matching  $M$ , the *auxiliary digraph*  $\tilde{G}_M$  is obtained from  $G$  as follows. Orient each edge in  $G$  from  $V^+$  to  $V^-$ . For each  $(\pi, \sigma) \in M$ , add a directed edge from  $\sigma$  to  $\pi$ . For  $\pi' \in \partial^+ M$  and  $\pi'' \in \text{cl}^+(\partial^+ M) \setminus \partial^+ M$ , add a directed edge from  $\pi'$  to  $\pi''$  if  $\partial^+ M \cup \{\pi''\} \setminus \{\pi'\}$  is independent in  $\mathbf{M}^+$ . For  $\sigma' \in \text{cl}^-(\partial^- M) \setminus \partial^- M$  and  $\sigma'' \in \partial^- M$ , add a directed edge from  $\sigma'$  to  $\sigma''$  if  $\partial^- M \cup \{\sigma'\} \setminus \{\sigma''\}$  is independent in  $\mathbf{M}^-$ . Let  $S := V^+ \setminus \text{cl}^+(\partial^+ M)$  and  $T := V^- \setminus \text{cl}^-(\partial^- M)$ .

**Lemma 2.3** (see [10, Lemma 2.3.32, Theorem 2.3.33, Lemma 2.3.35]). (1) *An independent matching  $M$  is maximum if and only if there is no directed path in  $\tilde{G}_M$  from  $S$  to  $T$ .*

(2) *A cover  $(H, K)$  is minimum if and only if it is represented as  $(H, K) = (V^+ \setminus C, V^- \cap C)$  for a vertex subset  $C$  such that  $S \subseteq C$ ,  $T \cap C = \emptyset$ , and no edge in  $\tilde{G}_M$  goes out from  $C$ .*

(3) *For a maximum independent matching  $M$  and a minimum cover  $(H, K)$ , it holds that  $\rho^+(H) = |H \cap \partial^+ M|$  and  $\rho^-(K) = |K \cap \partial^- M|$ .*

The following algorithm to find a maximum independent matching is due to Tomizawa and Iri [12]; see [10, p. 89].

**Algorithm to find a maximum independent matching:**

0.  $M := \emptyset$ .
1. Construct  $\tilde{G}_M$ . Find a shortest path  $P$  from  $S$  to  $T$  in  $\tilde{G}_M$ .
2. If such a path  $P$  does not exist, then  $M$  is a maximum independent matching; stop.
3. Let  $E_P$  be the set of edges  $(\pi, \sigma)$  in  $E$  such that  $(\pi, \sigma)$  or  $(\sigma, \pi)$  belongs to  $P$ . Let  $M := (M \setminus E_P) \cup (E_P \setminus M)$ , and go to 1.

Since  $P$  is shortest,  $M$  is always an independent matching [10, Lemma 2.3.31]. The size of  $M$  increases by one in each iteration. Thus the algorithm terminates after at most  $|E|$  iterations. The algorithm needs some matroid oracle to construct  $\tilde{G}_M$ . In our case,  $\mathbf{M}^+$  and  $\mathbf{M}^-$  are matroids of linear independence of vectors. Then  $\text{cl}^+(\partial^+ M)$  and  $\text{cl}^-(\partial^- M)$  as well as the edges to be added for  $\tilde{G}_M$  are calculated by Gaussian elimination.

### 3 Maximum stable subspace problem and DM-decomposition

In this section, we introduce DM-decomposition of partitioned matrix. Our approach takes a form different from the original approach by Ito, Iwata, and Murota [5]. It turns out (see Remark 3.3) that both approaches yield the same definition.

We first formulate a vector-space generalization of the stable set problem on a bipartite graph. Let  $U$  and  $V$  be finite dimensional vector spaces over a field  $\mathbf{F}$ . Let  $A : U \times V \rightarrow \mathbf{F}$  be a bilinear form. Let  $\mathcal{U}$  and  $\mathcal{V}$  be the lattices of all vector subspaces of  $U$  and of  $V$ , respectively. Let  $\mathcal{L}$  and  $\mathcal{M}$  be sublattices of  $\mathcal{U}$  and of  $\mathcal{V}$ , respectively. For subspace  $(X, Y) \in \mathcal{L} \times \mathcal{M}$ , let  $A(X, Y)$  denote the image of  $(X, Y)$  by  $A$ :

$$A(X, Y) = \{A(x, y) \mid x \in X, y \in Y\}.$$

Then either  $A(X, Y) = \{0\}$  or  $\mathbf{F}$ . A subspace  $(X, Y) \in \mathcal{L} \times \mathcal{M}$  is said to be *stable* if  $A(X, Y) = \{0\}$ . The *maximum stable subspace problem (MSSP)* on  $(\mathcal{L}, \mathcal{M}, A)$  is formulated as follows:

**MSSP:** Find a stable subspace  $(X, Y) \in \mathcal{L} \times \mathcal{M}$  such that  $\dim X + \dim Y$  is maximum.

A stable subspace  $(X, Y)$  is said to be *maximum* if  $\dim X + \dim Y$  is maximum among all stable subspaces.

**Lemma 3.1.** *Let  $(X, Y)$  and  $(X', Y')$  be stable subspaces.*

- (1) *Both  $(X \cap X', Y + Y')$  and  $(X + X', Y \cap Y')$  are stable.*
- (2) *If both  $(X, Y)$  and  $(X', Y')$  are maximum, then both  $(X \cap X', Y + Y')$  and  $(X + X', Y \cap Y')$  are maximum.*

*Proof.* (1). By  $A(X, Y \cap Y') = A(X', Y \cap Y') = \{0\}$ , we have  $A(X + X', Y \cap Y') = A(X, Y \cap Y') + A(X', Y \cap Y') = \{0\}$ . (2) follows from (1) and (2.1).  $\square$

A canonical block-triangular matrix representation of  $A$  is obtained from the family  $\mathcal{S}_{\max} \subseteq \mathcal{L} \times \mathcal{M}$  of maximum stable subspaces. Define the partial order  $\preceq$  on  $\mathcal{S}_{\max}$  by:  $(X, Y) \preceq (X', Y')$  if  $X \subseteq X'$  (and  $Y \supseteq Y'$ ). Then  $\mathcal{S}_{\max}$  is isomorphic to a sublattice of  $\mathcal{L}$ , and is a modular lattice. Consider a maximal chain  $(X^0, Y^0) \prec (X^1, Y^1) \prec \cdots \prec (X^h, Y^h)$  of  $\mathcal{S}_{\max}$ . Let  $i_k := \dim X^k$  and  $j_k := \dim Y^k$  for  $k = 0, 1, 2, \dots, h$ . Then  $0 \leq i_0 < i_1 < i_2 < \cdots < i_h \leq n$ , and  $m \geq j_0 > j_1 > \cdots > j_h \geq 0$ . Since  $i_k + j_k$  is the dimension of maximum stable subspaces, it holds that

$$i_{k+1} - i_k = j_k - j_{k+1} \quad (k = 0, 1, 2, \dots, h-1).$$

Let  $\{e_1, e_2, \dots, e_n\}$  be a basis of  $U$  such that  $\{e_1, e_2, \dots, e_{i_k}\}$  is a basis of  $X^k$  for  $k = 0, 1, 2, \dots, h$ . Also, let  $\{f_1, f_2, \dots, f_m\}$  be a basis of  $V$  such that  $\{f_1, f_2, \dots, f_{j_k}\}$  is a basis of  $Y^k$  for  $k = 0, 1, 2, \dots, h$ .

Let  $a_{ij} := A(e_{n+1-i}, f_j)$  for  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, m$ . Consider  $n \times m$  matrix  $A_{\text{DM}} = (a_{ij})$ , which is the matrix representation of  $A$  with respect to the basis  $\{e_i\}$  and  $\{f_j\}$ . For  $k = 0, 1, 2, \dots, h$ , let  $D_k$  denote the submatrix of  $A_{\text{DM}}$  consisting of  $A(e_i, f_j)$  for  $i_{k-1} < i \leq i_k$  and  $j_k < j \leq j_{k-1}$ , where  $i_{-1} := 0$ ,  $j_{-1} := m$ . Let  $D_\infty$  denote the submatrix consisting of  $A(e_i, f_j)$  for  $i_h < i \leq n$  and  $1 \leq j < j_h$ . Notice that  $D_0$  and  $D_\infty$  may be empty matrices. Since  $A(e_i, f_j) = 0$  for  $i \leq i_k, j \leq j_l$  with  $k \leq l$ , the matrix representation  $A_{\text{DM}}$  of  $A$  is in a block-triangular form as follows:

$$A_{\text{DM}} = \begin{pmatrix} D_\infty & & & & \\ O & D_h & & & \\ O & O & \ddots & & \\ \vdots & \vdots & \ddots & D_1 & \\ O & O & \dots & O & D_0 \end{pmatrix}.$$

This matrix representation  $A_{\text{DM}}$  of bilinear form  $A$  is called the *DM-representation with respect to  $(\mathcal{L}, \mathcal{M})$* . Here the diagonal block  $D_k$  is a square matrix of size  $d_k := i_{k+1} - i_k = j_k - j_{k+1}$  for  $k = 1, 2, \dots, h$ . In the case of  $V \in \mathcal{M}$ ,  $D_0$  is the empty matrix if and only if  $(X^0, Y^0) = (0, V)$ , i.e.,  $i_k + j_k = m$ . Consequently, if  $D_0$  is nonempty, then  $m < i_0 + j_0$  and the column size of  $D_0$  is less than the row size. Similarly, in the case of  $U \in \mathcal{L}$ , the row size of  $D_\infty$  is less than its column size when  $D_\infty$  is nonempty.

Although there are degrees of freedom in defining the entries of  $A_{\text{DM}}$ , its block structure is uniquely determined in the following sense. Obviously the sizes of  $D_0$  and  $D_\infty$  are independent of the choice of a maximal chain. Moreover the size sequence  $(d_1, d_2, \dots, d_h)$  is uniquely-determined up to permutation. This is a consequence of the Jordan-Hölder theorem. See [5] or [10, Section 4.8] for more details on the uniqueness.

Let us return to the case of partitioned matrices. Let  $A = (A_{\alpha\beta})$  be an  $n \times m$  partitioned matrix of the type  $(n_1, n_2, \dots, n_\mu; m_1, m_2, \dots, m_\nu)$ , where the entries of  $A$  are numbers in a field  $\mathbf{F}$ . Let  $U = \mathbf{F}^n$  and  $V = \mathbf{F}^m$ , where vectors in  $U$  and  $V$  are treated as column vectors. Regard  $A$  as a bilinear form on  $U \times V$  by

$$(x, y) \mapsto x^\top Ay.$$

For  $\alpha = 1, 2, \dots, \mu$ , let  $U_\alpha$  be the subspace of  $U$  consisting of  $u = (u_1, u_2, \dots, u_n)^\top$  such that  $u_i = 0$  for  $i \leq \sum_{\kappa=1}^{\alpha-1} n_\kappa$  or  $i > \sum_{\kappa=1}^\alpha n_\kappa$ . Also, for  $\beta = 1, 2, \dots, \nu$ , let  $V_\beta$  be the subspace of  $V$  consisting of  $v = (v_1, v_2, \dots, v_m)^\top$  such that  $v_j = 0$  for  $j \leq \sum_{\kappa=1}^{\beta-1} m_\kappa$  or  $j > \sum_{\kappa=1}^\beta m_\kappa$ . Then  $U = U_1 \oplus U_2 \oplus \dots \oplus U_\mu$  and  $V = V_1 \oplus V_2 \oplus \dots \oplus V_\nu$ . Let  $\mathcal{L}$  be the lattice of all subspaces  $X$  of  $U$  represented as  $X = X_1 \oplus X_2 \oplus \dots \oplus X_\mu$ , where  $X_\alpha$  is a subspace of  $U_\alpha$  for  $\alpha = 1, 2, \dots, \mu$ . Let  $\mathcal{M}$  be the lattice of all subspaces  $Y$  of  $V$  represented as  $Y = Y_1 \oplus Y_2 \oplus \dots \oplus Y_\nu$ , where  $Y_\beta$  is a subspace of  $V_\beta$  for  $\beta = 1, 2, \dots, \nu$ . Such a representation is unique, where  $X_\alpha = X \cap U_\alpha$  and  $Y_\beta = Y \cap V_\beta$ . Now  $A_{\alpha\beta}$  is viewed as a bilinear form on  $U_\alpha \times V_\beta$ . Then the subspace  $(X, Y) \in \mathcal{L} \times \mathcal{M}$  is stable if and only if

$$A_{\alpha\beta}(X_\alpha, Y_\beta) = \{0\}$$

for  $\alpha = 1, 2, \dots, \mu$  and  $\beta = 1, 2, \dots, \nu$ .

Consider the DM-representation  $A_{\text{DM}}$  of  $A$  with respect to  $(\mathcal{L}, \mathcal{M})$ . As above, choose a maximal chain  $(X^0, Y^0) \prec (X^1, Y^1) \prec \cdots \prec (X^h, Y^h)$  and bases  $\{e_1, e_2, \dots, e_n\}$  and  $\{f_1, f_2, \dots, f_m\}$ , where these bases are chosen so that  $\{e_1, e_2, \dots, e_n\} \cap U_\alpha$  is a basis of  $U_\alpha$  for  $\alpha = 1, 2, \dots, \mu$ , and  $\{f_1, f_2, \dots, f_m\} \cap V_\beta$  is a basis of  $V_\beta$  for  $\beta = 1, 2, \dots, \nu$ .

Let  $E := (e_n \ e_{n-1} \ \cdots \ e_1)$  be a nonsingular  $n \times n$  matrix of column vectors  $e_i$ , and let  $F := (f_1 \ f_2 \ \cdots \ f_m)$  be a nonsingular  $m \times m$  matrix of column vectors  $f_j$ . Then  $A_{\text{DM}}$  is given by  $E^\top A F$ , and is called the *DM-decomposition* of  $A$ . Notice that  $A \mapsto E^\top A F$  is an admissible transformation in the introduction. Indeed,  $E$  and  $F$  are represented as

$$E = \begin{pmatrix} E_1 & O & \cdots & O \\ O & E_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & E_\mu \end{pmatrix} P, \quad F = \begin{pmatrix} F_1 & O & \cdots & O \\ O & F_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & F_\nu \end{pmatrix} Q,$$

where  $E_\alpha$  is a nonsingular  $n_\alpha \times n_\alpha$  matrix,  $F_\beta$  is a nonsingular  $m_\beta \times m_\beta$  matrix, and  $P$  and  $Q$  are permutation matrices of sizes  $n$  and  $m$ , respectively.

**Remark 3.2** (Submodularity). Let  $\mathcal{S} \subseteq \mathcal{L} \times \mathcal{M}$  be the family of all stable subspaces. By Lemma 3.1 (1),  $\mathcal{S}$  is a modular lattice with join  $(X, Y) \vee (X', Y') = (X + X', Y \cap Y')$  and meet  $(X, Y) \wedge (X', Y') = (X \cap X', Y + Y')$ . Then (MSSP) is viewed as a *modular function* maximization over  $\mathcal{S}$ . Indeed, by (2.1), the objective function  $v(X, Y) := \dim X + \dim Y$  of (MSSP) satisfies the modular equality

$$v(X, Y) + v(X', Y') = v((X, Y) \wedge (X', Y')) + v((X, Y) \vee (X', Y')) \quad ((X, Y), (X', Y') \in \mathcal{S}).$$

Also (MSSP) can be formulated as a supermodular function maximization on the modular lattice  $\mathcal{M}$ . For  $Y \in \mathcal{M}$ , the family of  $X \in \mathcal{L}$  with  $(X, Y) \in \mathcal{S}$  forms a sublattice of  $\mathcal{L}$ . Let  $Y^\perp$  denote the maximum of this sublattice. Then (MSSP) is equivalent to:

**MSSP'**: Find  $Y \in \mathcal{M}$  such that  $\dim Y + \dim Y^\perp$  is maximum.

This is a supermodular function maximization over  $\mathcal{M}$ . Indeed, the function  $\gamma$  on  $\mathcal{M}$  defined by  $Y \mapsto \dim Y + \dim Y^\perp$  is *supermodular*:

$$\gamma(Y) + \gamma(Y') \leq \gamma(Y + Y') + \gamma(Y \cap Y') \quad (Y, Y' \in \mathcal{M}).$$

This immediately follows from (2.1) and  $Y^\perp + Y'^\perp \subseteq (Y \cap Y')^\perp$  and  $Y^\perp \cap Y'^\perp = (Y + Y')^\perp$ .

**Remark 3.3** (Relation to the original approach). In the case of a partitioned matrix, the function  $\gamma$  is equal to  $n$  minus the *surplus function* by Ito, Iwata, and Murota [5], where they defined the DM-decomposition from a maximal chain of the minimizer set of the surplus function. In their formulation, a partitioned matrix  $A$  is viewed as a linear map  $V \rightarrow U^* = U_1^* \oplus U_2^* \oplus \cdots \oplus U_\mu^*$ . The surplus function  $p$  on  $\mathcal{M}$  is defined by  $Y \mapsto \sum_\alpha \dim \text{Pr}_\alpha A(Y) - \dim Y$ , where  $\text{Pr}_\alpha : U^* \rightarrow U_\alpha^*$  is the projection. Observe that  $(Y^\perp)_\alpha$  is the orthogonal space of  $\text{Pr}_\alpha A(Y)$ . Thus  $\dim Y^\perp = \sum_\alpha \dim (Y^\perp)_\alpha = \sum_\alpha n_\alpha - \dim \text{Pr}_\alpha A(Y)$ , and  $\gamma = n - p$ . Consequently, our definition of the DM-decomposition is the same as the one in [5].

**Remark 3.4** (VCSP perspective). We see in the next section that (MSSP) for our special case is solvable in polynomial time, though we do not know whether (MSSP) is

tractable in general. We here remark another nontrivial tractable case. Let  $\mathcal{U}_\alpha$  and  $\mathcal{V}_\beta$  be the lattices of all vector subspaces of  $U_\alpha$  and of  $V_\beta$ , respectively. Then  $\mathcal{L}$  is isomorphic to  $\mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_\mu$ , and  $\mathcal{M}$  is isomorphic to  $\mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_\nu$ . Consider the case where  $\mathbf{F}$  is a finite field, and  $|\mathcal{U}_\alpha|$  and  $|\mathcal{V}_\beta|$  are constants for  $\alpha, \beta$ . An example is:  $\mathbf{F} = \text{GF}(2)$ ,  $n_\alpha = m_\beta = 2$ , and  $|\mathcal{U}_\alpha| = |\mathcal{V}_\beta| = 5$  for each  $\alpha, \beta$ . In this case, (MSSP) is viewed as a *valued constraint satisfaction problem (VCSP)*; see [7]. Define  $S_{\alpha\beta} : \mathcal{U}_\alpha \times \mathcal{V}_\beta \rightarrow \{0, -\infty\}$  by  $S_{\alpha\beta}(X_\alpha, Y_\beta) := 0$  if  $A_{\alpha\beta}(X_\alpha, Y_\beta) = \{0\}$  and  $-\infty$  otherwise. Then (MSSP) can be formulated as

$$\begin{aligned} \text{Max.} \quad & \sum_{\alpha} \dim X_\alpha + \sum_{\beta} \dim Y_\beta + \sum_{\alpha, \beta} S_{\alpha\beta}(X_\beta, Y_\beta) \\ \text{s.t.} \quad & (X_1, X_2, \dots, X_\mu) \in \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_\mu, \\ & (Y_1, Y_2, \dots, Y_\nu) \in \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_\nu. \end{aligned}$$

This is a binary VCSP, where the input is the table of all function values of  $\dim$  and  $S_{\alpha\beta}$ , and the total size is  $O(nm)$ . By Remark 3.2, this is a submodular VCSP, and hence it admits a *fractional polymorphism* corresponding to the submodularity inequality. Therefore, by the result of [7], the *basic LP relaxation* is exact, and solves (MSSP). Computing the DM-decomposition in this setting would be interesting and needs more arguments, which is left to a future work.

**Remark 3.5** (Rank of  $A$ ). Let  $v^*$  be the optimal value of (MSSP). The quantity  $n + m - v^*$  is an upper bound of the rank of  $A$ . Of course, this bound is far from being tight. A natural situation for this bound to be effective is the case where entries of distinct submatrices have no algebraic relations. A *generic partitioned matrix* [6] is a notion to capture this situation, and is a partitioned matrix  $(A_{\alpha\beta})$  (over a field  $\mathbf{F}$ ) such that  $A_{\alpha\beta}$  is represented as  $t_{\alpha\beta}B_{\alpha\beta}$ , where  $B_{\alpha\beta}$  is a matrix over a subfield  $\mathbf{K}$  of  $\mathbf{F}$  and elements  $t_{\alpha\beta}$  are algebraically independent over  $\mathbf{K}$ . For a multilayered mixed matrix  $A$  (a generic partitioned matrix  $A$  of type  $(n_1, n_2, \dots, n_\mu; 1, 1, \dots, 1)$ ), the CCF theory [10, 11] implies that the bound  $n + m - v^*$  is equal to  $\text{rank } A$ . Iwata and Murota [6] proved that this equality holds for any generic partitioned matrix with  $n_\alpha \leq 2, m_\beta \leq 2$ . They also presented an example of a  $6 \times 6$  generic partitioned matrix of type  $(3, 3; 2, 2, 2)$  not having this property. This example consists of rank-2 submatrices. It is a natural question whether this rank formula holds for our rank-1 case.

## 4 DM-decomposition of a partitioned matrix with rank-1 blocks

Let  $A = (A_{\alpha\beta})$  be an  $n \times m$  partitioned matrix of type  $(n_1, n_2, \dots, n_\mu; m_1, m_2, \dots, m_\nu)$ . Suppose now that

**(rank-1 condition)**  $\text{rank } A_{\alpha\beta} \leq 1$  for  $\alpha = 1, 2, \dots, \mu, \beta = 1, 2, \dots, \nu$ .

In the following, index  $\alpha$  ranges over  $1, 2, \dots, \mu$  and  $\beta$  ranges over  $1, 2, \dots, \nu$ .

### 4.1 Reduction

Here we show that (MSSP) under the rank-1 condition is reduced to an independent matching problem. The reduction is based on an intuitively obvious fact (Lemmas 4.2



and 4.3) that for any stable subspace  $(X, Y)$ , the component  $X_\alpha$  (resp.  $Y_\beta$ ) belongs to the intersection of  $\ker A_{\alpha\beta}^\top$  (resp.  $\ker A_{\alpha\beta}$ ) for several  $\beta$  (resp.  $\alpha$ ).

Let  $\Pi_\alpha$  be the set of hyperplanes  $\pi$  in  $U_\alpha$  such that  $\pi = \ker A_{\alpha\beta}^\top$  for some  $\beta$  (with  $\text{rank } A_{\alpha\beta} = 1$ ). Also, let  $\Sigma_\beta$  be the set of hyperplanes  $\sigma$  in  $V_\beta$  such that  $\sigma = \ker A_{\alpha\beta}$  for some  $\alpha$ . Let  $\Pi$  be the disjoint union of  $\Pi_\alpha$  over  $\alpha$ , and let  $\Sigma$  be the disjoint union of  $\Sigma_\beta$  over  $\beta$ . Define bipartite graph  $G = (\Pi, \Sigma, E)$  on vertex set  $\Pi \cup \Sigma$ , where for each  $\alpha, \beta$ ,

$$(\pi, \sigma) \in \Pi_\alpha \times \Sigma_\beta \text{ is an edge in } E \text{ if and only if } \pi = \ker A_{\alpha\beta}^\top \text{ and } \sigma = \ker A_{\alpha\beta}.$$

For  $H \subseteq \Pi$ , define subspace  $X(H) = X(H)_1 \oplus X(H)_2 \oplus \cdots \oplus X(H)_\mu$  of  $\mathcal{L}$  by  $X(H)_\alpha :=$  the intersection of all hyperplanes in  $H \cap \Pi_\alpha$ . Similarly, for  $K \subseteq \Sigma$ , define subspace  $Y(K) = Y(K)_1 \oplus Y(K)_2 \oplus \cdots \oplus Y(K)_\nu$  of  $\mathcal{M}$  by  $Y(K)_\beta :=$  the intersection of all hyperplanes in  $K \cap \Sigma_\beta$ .

Consider matroids  $\mathbf{M}(\Pi_\alpha)$  and  $\mathbf{M}(\Sigma_\beta)$  defined in Section 2.2. Let  $\mathbf{M}(\Pi)$  be the direct sum of matroids  $\mathbf{M}(\Pi_\alpha)$  over  $\alpha$ , and let  $\mathbf{M}(\Sigma)$  be the direct sum of matroids  $\mathbf{M}(\Sigma_\beta)$  over  $\beta$ . Consider the independent matching problem on  $(\mathbf{M}(\Pi), \mathbf{M}(\Sigma), G)$ . Then the maximum stable subspace is obtained from a minimum cover in  $(\mathbf{M}(\Pi), \mathbf{M}(\Sigma), G)$  as follows.

**Theorem 4.1.** *Let  $A$  be an  $n \times m$  partitioned matrix satisfying the rank-1 condition.*

- (1) *A stable subspace  $(X, Y)$  is maximum if and only if  $(X, Y)$  is represented as  $(X(H), Y(K))$  for a minimum cover  $(H, K)$  in  $(\mathbf{M}(\Pi), \mathbf{M}(\Sigma), G)$ .*
- (2) *The maximum dimension of a stable subspace is equal to  $n + m$  minus the maximum size of an independent matching in  $(\mathbf{M}(\Pi), \mathbf{M}(\Sigma), G)$ .*

The rest of this subsection is devoted to proving this theorem. Let  $A = (A_{\alpha\beta})$  be a partitioned matrix satisfying the rank-1 condition. The following property of a rank-1 matrix is essential in our reduction.

**Lemma 4.2.** *For subspaces  $X_\alpha \subseteq U_\alpha$  and  $Y_\beta \subseteq V_\beta$ , if  $A_{\alpha\beta}(X_\alpha, Y_\beta) = \{0\}$ , then  $X_\alpha \subseteq \ker A_{\alpha\beta}^\top$  or  $Y_\beta \subseteq \ker A_{\alpha\beta}$ .*

*Proof.* Suppose to the contrary that  $A_{\alpha\beta}(X_\alpha, Y_\beta) = \{0\}$ ,  $X_\alpha \not\subseteq \ker A_{\alpha\beta}^\top$ , and  $Y_\beta \not\subseteq \ker A_{\alpha\beta}$ . Then  $0 \neq X_\alpha \neq U_\alpha$ ,  $0 \neq Y_\beta \neq V_\beta$ ,  $A_{\alpha\beta}(X_\alpha, V_\beta) \neq \{0\}$ , and  $A_{\alpha\beta}(U_\alpha, Y_\beta) \neq \{0\}$ . There are nonzero vectors  $u \in X_\alpha, u' \in U_\alpha \setminus X_\alpha, v \in Y_\beta, v' \in V_\beta \setminus Y_\beta$  such that  $A_{\alpha\beta}(u, v') \neq 0$ ,  $A_{\alpha\beta}(v, u') \neq 0$ , and  $A_{\alpha\beta}(u, v) = 0$ . Thus

$$\det \begin{pmatrix} A_{\alpha\beta}(u, v') & A_{\alpha\beta}(u', v') \\ A_{\alpha\beta}(u, v) & A_{\alpha\beta}(u', v) \end{pmatrix} = \det \begin{pmatrix} A_{\alpha\beta}(u, v') & A_{\alpha\beta}(u', v') \\ 0 & A_{\alpha\beta}(u', v) \end{pmatrix} \neq 0.$$

This means that a  $2 \times 2$  submatrix of a matrix obtained from  $A_{\alpha\beta}$  by change of basis has nonzero determinant, and hence  $\text{rank } A_{\alpha\beta} \geq 2$ . This contradicts  $\text{rank } A_{\alpha\beta} \leq 1$ .  $\square$

For a subspace  $(X, Y)$  in  $\mathcal{L} \times \mathcal{M}$ , let  $\Pi_{\alpha, X}$  denote the set of hyperplanes in  $\Pi_\alpha$  containing  $X_\alpha$ , and let  $\Sigma_{\beta, Y}$  denote the set of hyperplanes in  $\Sigma_\beta$  containing  $Y_\beta$ . Let  $\Pi_X$  be the union of  $\Pi_{\alpha, X}$  over  $\alpha$ , and let  $\Sigma_Y$  be the union of  $\Sigma_{\beta, Y}$  over  $\beta$ .

**Lemma 4.3.** (1) *If  $(H, K) \in \Pi \times \Sigma$  is a cover, then  $(X(H), Y(K))$  is a stable subspace.*

(2) If  $(X, Y)$  is a stable subspace, then  $(\Pi_X, \Sigma_Y)$  is a cover.

(3) If  $(X, Y)$  is a maximum stable subspace, then  $(X, Y) = (X(\Pi_X), Y(\Sigma_Y))$ .

*Proof.* (1). Suppose that  $(X, Y) = (X(H), Y(K))$  is not stable. Then  $A_{\alpha\beta}(X_\alpha, Y_\beta) \neq \{0\}$  for some  $\alpha, \beta$ . This implies that  $X_\alpha \not\subseteq \ker A_{\alpha\beta}^\top$  and  $Y_\beta \not\subseteq \ker A_{\alpha\beta}$ . Therefore  $\Pi_X$  does not contain  $\ker A_{\alpha\beta}^\top$  and  $\Sigma_Y$  does not contain  $\ker A_{\alpha\beta}$ . By  $H \subseteq \Pi_X$  and  $K \subseteq \Sigma_Y$ , the edge between  $\ker A_{\alpha\beta}^\top$  and  $\ker A_{\alpha\beta}$  does not meet  $H \cup K$ . Namely  $(H, K)$  is not a cover.

(2). Suppose that  $(X, Y)$  is stable. Consider arbitrary  $\alpha, \beta$ . Now  $A_{\alpha\beta}(X_\alpha, Y_\beta) = \{0\}$ . By Lemma 4.2, it holds that  $X_\alpha \subseteq \ker A_{\alpha\beta}^\top$  or  $Y_\beta \subseteq \ker A_{\alpha\beta}$ . This means that the endpoints of edge joining  $\ker A_{\alpha\beta}^\top$  and  $\ker A_{\alpha\beta}$  meet  $\Pi_{\alpha, X} \cup \Sigma_{\beta, Y}$ .

(3). Suppose that  $(X, Y)$  is a maximum stable subspace. By (1) and (2), subspace  $(X(\Pi_X), Y(\Sigma_Y))$  is also stable. By  $X \subseteq X(\Pi_X)$ ,  $Y \subseteq Y(\Sigma_Y)$ , and the maximality, it must hold that  $(X, Y) = (X(\Pi_X), Y(\Sigma_Y))$ .  $\square$

Thus the problem (MSSP) under the rank-1 condition is equivalent to:

**MC:** Find a cover  $(H, K)$  such that  $\dim X(H) + \dim Y(K)$  is maximum.

Now  $\dim X(H)$  is equal to  $n$  minus the rank of  $H$  in  $\mathbf{M}(\Pi)$ , and  $\dim Y(K)$  is equal to  $m$  minus the rank of  $K$  in  $\mathbf{M}(\Sigma)$ . Namely, (MC) is nothing but the minimum cover problem dual to the independent matching problem on  $(\mathbf{M}(\Pi), \mathbf{M}(\Sigma), G)$ . This proves Theorem 4.1.

## 4.2 Algorithm

Here we present an algorithm to compute the DM-decomposition  $A_{\text{DM}}$ . Let  $M$  be a maximum independent matching, which is obtained by the algorithm in Section 2.2 with  $V^+ = \Pi$ ,  $V^- = \Sigma$ ,  $\mathbf{M}^+ = \mathbf{M}(\Pi)$ , and  $\mathbf{M}^- = \mathbf{M}(\Sigma)$ . From  $\tilde{G}_M$ , we are going to construct a compact representation of  $\mathcal{S}_{\max}$ . Let  $C_0$  be the set of vertices  $v$  having a path from  $S$  to  $v$ , and let  $C_\infty$  the set of vertices  $v$  having a path from  $v$  to  $T$ . Let  $H_0, H_\infty, K_0, K_\infty$  be the subsets of vertices defined by

$$\begin{aligned} H_0 &:= C_0 \cap \partial^+ M, & K_0 &:= C_0 \cap \partial^- M, \\ H_\infty &:= C_\infty \cap \partial^+ M, & K_\infty &:= C_\infty \cap \partial^- M. \end{aligned}$$

Let  $\tilde{G}'_M$  be the digraph obtained from  $\tilde{G}_M$  by deleting  $C_0$  and  $C_\infty$ . Consider the strongly connected component decomposition of  $\tilde{G}'_M$ . Let  $h$  be the number of components meeting  $\partial^+ M \cup \partial^- M$ . Consider a partition  $\{H_0, H_1, H_2, \dots, H_h, H_\infty\}$  of  $\partial^+ M$  such that  $\pi$  and  $\pi'$  belong to  $H_k$  ( $1 \leq k \leq h$ ) if and only if  $\pi$  and  $\pi'$  belong to the same component. Accordingly, consider a partition  $\{K_0, K_1, K_2, \dots, K_h, K_\infty\}$  of  $\partial^- M$  such that  $H_k$  is matched to  $K_k$  by  $M$  for  $k = 1, 2, \dots, h$  (or  $K_k$  belongs to the same component as  $H_k$ ). Define a partial order  $\preceq$  on  $\mathcal{P} := \{1, 2, \dots, h\}$  such that  $k \preceq l$  if and only if there is a directed path in  $\tilde{G}'_M$  from  $H_l$  to  $H_k$ . For an ideal  $J \in \mathcal{J}(\mathcal{P})$ , define  $H_J \subseteq \Pi$  and  $K_J \subseteq \Sigma$  by

$$H_J := \bigcup \{H_k \mid k \in \mathcal{P} \cup \{\infty\} \setminus J\}, \quad K_J := \bigcup \{K_k \mid k \in J \cup \{0\}\}.$$

**Proposition 4.4.**  $\mathcal{J}(\mathcal{P})$  is isomorphic to  $\mathcal{S}_{\max}$ , where an isomorphism is given by

$$\mathcal{J}(\mathcal{P}) \ni J \mapsto (X(H_J), Y(K_J)). \quad (4.1)$$

In particular,  $\mathcal{S}_{\max}$  is isomorphic to a distributive sublattice of  $\mathcal{L}$ .

*Proof.* Let  $J \in \mathcal{J}(\mathcal{P})$ . Let  $C'$  be the set of vertices in  $\tilde{G}_M$  reachable from a vertex in  $\bigcup_{k \in J} H_k$ . Let  $C := C' \cup C_0$ . Then  $S \subseteq C$ ,  $T \cap C = \emptyset$ , and no edge goes out from  $C$ . Hence  $(\Pi \setminus C, \Sigma \cap C)$  is a minimum cover (Theorem 4.1 (2)). Thus  $(X(\Pi \setminus C), Y(\Sigma \cap C))$  is a maximum stable subspace (Theorem 4.1 (2)). By definition of  $C$  and  $J \in \mathcal{J}(\mathcal{P})$ , it holds  $\partial^+ M \setminus C = H_J$  and  $\partial^- M \cap C = K_J$ . Also the rank of  $\Pi \setminus C$  in  $\mathbf{M}(\Pi)$  is equal to  $|\partial^+ M \setminus C|$ , and the rank of  $\Sigma \cap C$  in  $\mathbf{M}(\Sigma)$  is equal to  $|\partial^- M \cap C|$  (Lemma 2.3 (3)). Thus  $X(H_J) = X(\Pi \setminus C)$ ,  $Y(K_J) = Y(\Sigma \cap C)$ , and  $(X(H_J), Y(K_J)) \in \mathcal{S}_{\max}$ .

Conversely, let  $(X, Y)$  be a maximum stable subspace. Then  $(\Pi_X, \Sigma_Y)$  is a minimum cover. By Lemma 2.3,  $(\Pi_X, \Sigma_Y) = (\Pi \setminus C, \Sigma \cap C)$  holds for some  $C$  such that  $S \subseteq C$ ,  $T \cap C = \emptyset$ , and  $(*)$  no edge goes out from  $C$ . Let  $J$  be the set of indices  $k \in \{1, 2, \dots, h\}$  such that  $H_k$  belongs to  $C$ . Then, by the property  $(*)$ ,  $J$  is an ideal of  $\mathcal{P}$ .  $\square$

Thus, from the strongly connected component decomposition of  $\tilde{G}'_M$ , we obtain a poset  $\mathcal{P}$  representing  $\mathcal{S}_{\max}$  as  $\mathcal{S}_{\max} \simeq \mathcal{J}(\mathcal{P})$ . Relabel  $\mathcal{P} = \{1, 2, \dots, h\}$  so that  $k \prec l$  implies  $k < l$  for  $k, l \in \mathcal{P}$ . Then  $\{1, 2, \dots, k\}$  is an ideal. For  $k = 0, 1, 2, \dots, h$ , define stable subspace  $(X^k, Y^k)$  by

$$X^k := X(H_{k+1} \cup H_{k+2} \cup \dots \cup H_h \cup H_\infty), \quad Y^k := Y(K_0 \cup K_1 \cup K_2 \cup \dots \cup K_k).$$

Then  $(X^0, Y^0) \prec (X^1, Y^1) \prec \dots \prec (X^h, Y^h)$  is a maximal chain in  $\mathcal{S}_{\max}$ .

Next we construct bases of  $U$  and of  $V$  to obtain  $A_{\text{DM}}$ . A hyperplane in  $\Pi_\alpha$  is identified with its normal vector, which is an  $n_\alpha$ -dimensional row vector. Similarly a hyperplane in  $\Sigma_\beta$  is identified with an  $m_\beta$ -dimensional row vector. Each submatrix  $A_{\alpha\beta}$  of rank 1 is represented as  $c\pi^\top\sigma$  for some  $\pi \in \Pi_\alpha$ ,  $\sigma \in \Sigma_\beta$ , and  $c \in \mathbf{F} \setminus \{0\}$ . Now  $\Pi_\alpha$  is a set of  $n_\alpha$ -dimensional row vectors, and  $\Sigma_\beta$  is a set of  $m_\beta$ -dimensional row vectors.  $\mathbf{M}(\Pi_\alpha)$  and  $\mathbf{M}(\Sigma_\beta)$  are matroids of linear independence of these vectors. For each  $\alpha$ , choose any set  $\tilde{\Pi}_\alpha$  of vectors such that  $(\partial^+ M \cap \Pi_\alpha) \cup \tilde{\Pi}_\alpha$  is a basis of (the dual space of)  $\mathbf{F}^{n_\alpha}$ , and add  $\tilde{\Pi}_\alpha$  to  $H_0$ . Let  $H := H_0 \cup H_1 \cup \dots \cup H_h \cup H_\infty$ . Similarly, for each  $\beta$ , choose any set  $\tilde{\Sigma}_\beta$  of vectors such that  $(\partial^- M \cap \Sigma_\beta) \cup \tilde{\Sigma}_\beta$  is a basis of (the dual space of)  $\mathbf{F}^{m_\beta}$ , and add  $\tilde{\Sigma}_\beta$  to  $K_\infty$ . Let  $K := K_0 \cup K_1 \cup \dots \cup K_h \cup K_\infty$ . Suppose that  $H = \{\pi_1, \pi_2, \dots, \pi_n\}$ , where indices are ordered as: if  $\pi_i \in H_k$ ,  $\pi_j \in H_l$ , and  $k < l$ , then  $i < j$ . Suppose that  $K = \{\sigma_1, \sigma_2, \dots, \sigma_m\}$ , where indices are ordered as: if  $\sigma_i \in K_k$ ,  $\sigma_j \in K_l$ , and  $k < l$ , then  $i < j$ .

Then  $\Pi_\alpha \cap H = \{\pi_{\alpha_1}, \pi_{\alpha_2}, \dots, \pi_{\alpha_{n_\alpha}}\}$  for  $\alpha_1 < \alpha_2 < \dots < \alpha_{n_\alpha}$  and  $\Sigma_\beta \cap K = \{\sigma_{\beta_1}, \sigma_{\beta_2}, \dots, \sigma_{\beta_{m_\beta}}\}$  for  $\beta_1 < \beta_2 < \dots < \beta_{m_\beta}$ . Let  $R_\alpha$  be a nonsingular  $n_\alpha \times n_\alpha$  matrix such that the  $\lambda$ th row vector is  $\pi_{\alpha_\lambda}$ , and let  $S_\beta$  be a nonsingular  $m_\beta \times m_\beta$  matrix such that the  $\lambda$ th row vector is  $\sigma_{\beta_\lambda}$ :

$$R_\alpha := \begin{pmatrix} \pi_{\alpha_1} \\ \pi_{\alpha_2} \\ \vdots \\ \pi_{\alpha_{n_\alpha}} \end{pmatrix} \quad S_\beta := \begin{pmatrix} \sigma_{\beta_1} \\ \sigma_{\beta_2} \\ \vdots \\ \sigma_{\beta_{m_\beta}} \end{pmatrix}. \quad (4.2)$$

Let  $E_\alpha$  be a nonsingular  $n_\alpha \times n_\alpha$  matrix such that  $R_\alpha E_\alpha$  is upper-triangular. Let  $e_{\alpha,\lambda}$  denote the  $\lambda$ th column vector of  $E_\alpha$ . Let  $F_\beta$  be a nonsingular  $m_\beta \times m_\beta$  matrix such that  $S_\beta F_\beta$  is lower-triangular. Let  $f_{\beta,\lambda}$  denote the  $\lambda$ th column vector of  $F_\beta$ . For  $i = 1, 2, \dots, n$ , define an  $n$ -dimensional vector  $e_i$  as follows. Suppose that  $\alpha_\lambda = i$ . For  $j$  with  $\sum_{k=0}^{\alpha-1} |H_k| < j \leq \sum_{k=0}^\alpha |H_k|$ , the  $j$ th component of  $e_i$  is equal to the  $(j - \sum_{k=0}^{\alpha-1} |H_k|)$ th component of  $e_{\alpha,\lambda}$ . All other components of  $e_i$  are defined to be zero. For  $j = 1, 2, \dots, m$ , define  $m$ -dimensional vector  $f_j$  as follows. Suppose that  $\beta_\lambda = j$ . For  $i$  with  $\sum_{k=0}^{\beta-1} |K_k| < i \leq \sum_{k=0}^\beta |K_k|$ , the  $i$ th component of  $f_j$  is equal to the  $(i - \sum_{k=0}^{\beta-1} |K_k|)$ th component of  $f_{\beta,\lambda}$ . All other components of  $f_j$  are defined to be zero.

Then the DM-decomposition  $A_{\text{DM}}$  of  $A$  is given by

$$A_{\text{DM}} = (e_n \ e_{n-1} \ \cdots \ e_1)^\top A (f_m \ f_{m-1} \ \cdots \ f_1),$$

which is verified by the next lemma.

**Lemma 4.5.** *For  $k = 0, 1, 2, \dots, h$ , the following hold:*

- (1)  $\{e_1, e_2, \dots, e_{i_k}\}$  is a basis of  $X^k$  with  $i_k = \sum_{l=0}^k |H_l|$ .
- (2)  $\{f_{j_k+1}, f_{j_k+2}, \dots, f_m\}$  is a basis of  $Y^k$  with  $j_k = \sum_{l=0}^k |K_l|$ .

*Proof.* (1) Suppose that  $\Pi_\alpha \cap H = \{\pi_{\alpha_1}, \pi_{\alpha_2}, \dots, \pi_{\alpha_{n_\alpha}}\}$  for  $\alpha_1 < \alpha_2 < \dots < \alpha_{n_\alpha}$ . Then  $\Pi_\alpha \cap (H_{k+1} \cup H_{k+2} \cup \dots \cup H_\infty) = \{\pi_{\alpha_\lambda}, \pi_{\alpha_{\lambda+1}}, \dots, \pi_{\alpha_{n_\alpha}}\}$  for the minimum  $\lambda$  with  $\sum_{l=0}^k |H_l| < \alpha_\lambda$ . Then  $(X^k)_\alpha := X(H_{k+1} \cup H_{k+2} \cup \dots \cup H_\infty)_\alpha$  is the intersection of hyperplanes  $\pi_{\alpha_\lambda}, \pi_{\alpha_{\lambda+1}}, \dots, \pi_{\alpha_{n_\alpha}}$ . Since  $R_\alpha E_\alpha$  is upper-triangular, all  $e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_{\lambda-1}}$  belong to  $(X^k)_\alpha$ , and span  $(X^k)_\alpha$ . Consequently, the statement holds.

(2) Suppose that  $\Sigma_\beta \cap K = \{\sigma_{\beta_1}, \sigma_{\beta_2}, \dots, \sigma_{\beta_{m_\beta}}\}$  for  $\beta_1 < \beta_2 < \dots < \beta_{m_\beta}$ . Then  $\Sigma_\beta \cap (K_0 \cup K_1 \cup \dots \cup K_k) = \{\sigma_{\beta_1}, \sigma_{\beta_2}, \dots, \sigma_{\beta_\lambda}\}$  for the maximum  $\lambda$  with  $\beta_\lambda \leq \sum_{l=0}^k |K_l|$ . Then  $(Y^k)_\beta := Y(K_0 \cup K_1 \cup \dots \cup K_k)_\beta$  is the intersection of hyperplanes  $\sigma_{\beta_1}, \sigma_{\beta_2}, \dots, \sigma_{\beta_\lambda}$ . Since  $S_\beta F_\beta$  is lower-triangular, all  $f_{\beta_{\lambda+1}}, f_{\beta_{\lambda+2}}, \dots, f_{\beta_{m_\beta}}$  belong to  $(Y^k)_\beta$ , and span  $(Y^k)_\beta$ .  $\square$

Finally we give a rough estimate of the time complexity to compute  $A_{\text{DM}}$ . Suppose that each submatrix  $A_{\alpha\beta}$  of rank 1 is given as an expression  $\pi^\top \sigma$  for row vectors  $\pi, \sigma$ . The bipartite graph  $G = (\Pi, \Sigma, E)$  has  $O(\mu\nu)$  vertices and  $O(\mu\nu)$  edges. Therefore the number of iterations of the independent matching algorithm is bounded by  $\mu\nu$ . In the construction of  $\tilde{G}_M$ , the edges added to  $\Pi_\alpha$  are identified by Gaussian elimination in  $O(n_\alpha^2 |\Pi_\alpha|)$  time. Consequently we can construct  $\tilde{G}_M$  in  $O(n^2 \nu + m^2 \mu)$  time. An augmenting path is found in  $O(\mu\nu)$  time. Thus a maximum independent matching  $M$  is obtained in  $O(\mu^2 \nu^3 n^2 + \mu^3 \nu^2 m^2) = O(n^4 m^3 + n^3 m^4)$  time. The poset  $\mathcal{P}$  is naturally obtained from the final  $\tilde{G}'_M$  by the strongly connected component decomposition, and a maximal chain is also naturally obtained. Matrices  $E_\alpha, F_\beta$  are obtained in  $O(n^3 + m^3)$  time. The matrix  $(e_n \ e_{n-1} \ \cdots \ e_1)^\top A (f_m \ f_{m-1} \ \cdots \ f_1)$  is calculated in  $O(nm^2 + n^2 m)$  time. The total time is  $O(n^4 m^3 + n^3 m^4)$ . This proves Theorem 1.1.

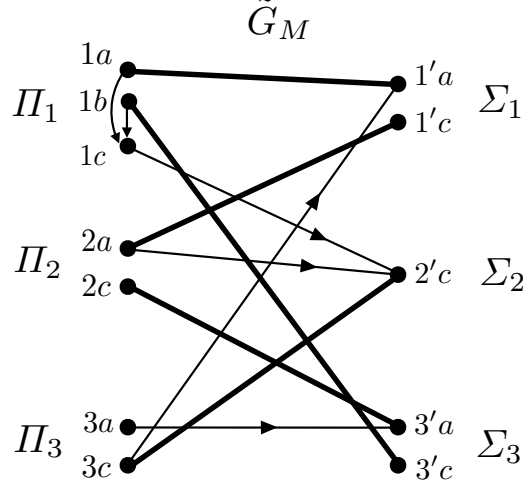


Figure 1: Auxiliary digraph  $\tilde{G}_M$  for a maximum independent matching  $M$

**Example.** Consider the following partitioned matrix of type  $(2, 2, 2; 2, 2, 2)$  over field  $\mathbf{F} = \text{GF}(2)$ :

$$A = \left( \begin{array}{cc|cc|cc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{array} \right)$$

This matrix satisfies the rank-1 condition. According to the above algorithm, the DM-decomposition  $A_{\text{DM}}$  is computed as follows. First we identify hyperplane sets  $\Pi_\alpha$  and  $\Sigma_\beta$ . Three hyperplanes (of normal vectors)  $(1 \ 0)$ ,  $(0 \ 1)$ , and  $(1 \ 1)$  in  $U_\alpha$  are denoted by  $\alpha a$ ,  $\alpha b$ , and  $\alpha c$ , respectively. Similarly, three hyperplanes  $(1 \ 0)$ ,  $(0 \ 1)$ , and  $(1 \ 1)$  in  $V_\beta$  are denoted by  $\beta' a$ ,  $\beta' b$ , and  $\beta' c$ , respectively. Then  $\Pi_1 = \{1a, 1b, 1c\}$ ,  $\Pi_2 = \{2a, 2c\}$ ,  $\Pi_3 = \{3a, 3c\}$ ,  $\Sigma_1 = \{1'a, 1'c\}$ ,  $\Sigma_2 = \{2'c\}$ , and  $\Sigma_3 = \{3'a, 3'c\}$ . Consider the independent matching problem on  $(\mathbf{M}(\Pi), \mathbf{M}(\Sigma), G)$ . Figure 1 depicts the auxiliary digraph  $\tilde{G}_M$  for a maximum independent matching  $M = \{(1a, 1'a), (1b, 3'c), (2a, 1'c), (2c, 3'a), (3c, 2'c)\}$ , where two directed edges corresponding to an edge in  $M$  is drawn by a single thick undirected edge. Now  $S = \{3a\}$ ,  $T = \emptyset$ ,  $C_0 = \{3a, 3'a, 2c\}$ , and  $C_\infty = \emptyset$ . There are three strongly connected components in  $\tilde{G}'_M$  meeting  $M$ . Then  $H_k$  and  $K_k$  ( $k = 0, 1, 2, 3, \infty$ ) are given as

$$\begin{aligned} H_\infty &= \emptyset, & K_\infty &= \emptyset, \\ H_3 &= \{1b\}, & K_3 &= \{3'c\}, \\ H_2 &= \{2a\}, & K_2 &= \{1'c\}, \\ H_1 &= \{1a, 3c\}, & K_1 &= \{1'a, 2'c\}, \\ H_0 &= \{2c\}, & K_0 &= \{3'a\}, \end{aligned}$$

where the partial order  $\preceq$  on  $\mathcal{P} = \{1, 2, 3\}$  given by

$$2 \succ 1 \prec 3.$$

Add  $3a$  to  $H_0$ . The elements in  $H = \bigcup_k H_k$  are ordered as  $2c, 3a, 1a, 3c, 2a, 1b$ . Add  $2'a$  to  $K_\infty$ . The elements in  $K = \bigcup_k K_k$  are ordered as  $3'a, 1'a, 2'c, 1'c, 3'c, 2'a$ . Matrices  $R_1, R_2, R_3, S_1, S_2, S_3$  are given by

$$\begin{aligned} R_1 &= \begin{pmatrix} 1a \\ 1b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & S_1 &= \begin{pmatrix} 1'a \\ 1'c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\ R_2 &= \begin{pmatrix} 2c \\ 2a \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, & S_2 &= \begin{pmatrix} 2'c \\ 2'a \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \\ R_3 &= \begin{pmatrix} 3a \\ 3c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & S_3 &= \begin{pmatrix} 3'a \\ 3'c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

Matrices  $E_1, E_2, E_3, F_1, F_2, F_3$  are given by

$$\begin{aligned} E_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & F_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ E_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & F_2 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \\ E_3 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & F_3 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Then the DM-decomposition  $A_{\text{DM}}$  is given by

$$\begin{aligned} A_{\text{DM}} &= \begin{pmatrix} 0 & & 1 & & & \\ & 1 & & 0 & & \\ & & 1 & & 0 & \\ & 0 & & & 1 & \\ & & 0 & 1 & & \\ & & 1 & 1 & & \end{pmatrix}^\top \left( \begin{array}{cc|cc|cc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{array} \right) \begin{pmatrix} & 0 & & 1 & & \\ & 1 & & 0 & & \\ & & 1 & & 1 & \\ 1 & & & 0 & & \\ & & & & 1 & \\ 1 & & & & & 0 \end{pmatrix} \\ &= \begin{pmatrix} \boxed{1} & 0 & 1 & 0 & 1 & \\ & \boxed{1} & 1 & 1 & 1 & \\ & & \boxed{1} & 1 & 0 & \\ & & & \boxed{1} & 1 & 0 \\ & & & & \boxed{1} & \\ & & & & & \boxed{1} \end{pmatrix}, \end{aligned}$$

where all empty entries are zero, and diagonal blocks  $D_0, D_1, D_2, D_3, D_\infty (= \emptyset)$  are indicated by dashed boxes.

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